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A two-parameter quantization of $sl(2/1)$ and its finite-dimensional representations

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Abstract. The Lie superalgebra $sl(2/1)$ is quantized in its non-standard simple root system, resulting in a two-parameter quantum superalgebra $U_{q_1,q_2}(sl(2/1))$. When the two parameters coincide, $U_{q_1,q_2}(sl(2/1))$ reduces to a one-parameter dependent $\mathbb{Z}_2$-graded Hopf algebra, which is algebraically equivalent, but coalgebraically inequivalent, to the standard $U_q(sl(2/1))$. The finite-dimensional irreducible representations of this two-parameter quantum superalgebra are explicitly constructed when both or one of q_1 and q_2 are considered as indeterminates, and cyclic representations are also obtained when both deformation parameters are roots of unity.

1. Introduction

It is well known that Lie superalgebras admit Weyl group inequivalent simple root systems [1, 2]. The simplest example is the rank-2 Lie superalgebra $sl(2/1)$. Its standard simple root system consists of one even and one odd simple root, while the non-standard one has both simple roots odd. It is merely a matter of convention which simple root system one chooses for a Lie superalgebra. However, in the quantization of Lie superalgebras, the simple generators play a special role, especially where the comultiplication is concerned, and it becomes unclear whether different choices of simple root systems will lead to equivalent $\mathbb{Z}_2$-graded Hopf algebras. This problem is interesting not only mathematically, but also for practical purposes. For example, if the quantum supergroups associated with different simple root systems are inequivalent, then there will exist different universal R-matrices, and it may also be possible to construct different spectral parameter-dependent solutions of the Yang–Baxter equation using their representation theory.

Another question is whether it is possible to carry out multiparameter quantizations of finite-dimensional Lie superalgebras when the non-standard simple root systems are used, such that the resultant quantum superalgebras admit non-trivial finite-dimensional representations when the deformation parameters are regarded as indeterminates, and reduce to quantum supergroups of Drinfeld–Jimbo [3, 4] type in a certain limit of the deformation parameters. There are many papers studying multiparameter quantizations [5–13], and a lot of interesting results have already been obtained. From these works, we can also see that it is a highly non-trivial matter to carry out multiparameter deformations for both Lie algebras and Lie superalgebras, especially when one requires that the resultant (super)algebras should depend on the extra parameters intrinsically. The only such multiparameter quantum superalgebras obtained so far are the two-parameter $sl(1/1)$ [11] and $osp(4/2)$ [12], and Lee's [13] $U_{q,s}(gl(m); n)$ which depends on a generic parameter q and a root of unity s.

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The difficulty in constructing interesting multiparameter quantum algebras lies in the fact that extra parameters often spoil the Serre relations, and this in turn rules out the possibility of the (super)algebra having non-trivial finite-dimensional representations. However, for some superalgebras in their non-standard simple root systems, the Serre relations appear much less stringent. In fact, in the case of $sl(2/1)$, no Serre relations are required at all in order to present the superalgebra by generators and relations. Therefore, there seems to be room to manoeuvre in searching for multiparameter deformations.

In this paper we consider the quantization of $sl(2/1)$ in its non-standard simple root system, and study the finite-dimensional representations of the resultant superalgebras. A two-parameter quantum superalgebra, $U_{q_1,q_2}(sl(2/1))$, is obtained, which has distinct algebraic structures from the standard quantum group $U_q(sl(2/1))$. When q_1 is equal to q_2, it reduces to a one-parameter quantum superalgebra, $U_{q,q}(sl(2/1))$, which admits comultiplication, counit and antipode, and hence has the structures of a $\mathbb{Z}_2$-graded Hopf algebra. We will show that this quantum supergroup is algebraically equivalent to, but coalgebraically different from, $U_q(sl(2/1))$. Due to the presence of the extra parameter in $U_{q_1,q_2}(sl(2/1))$, new features also arise in the representation theory, thus rendering it more interesting to study. We will classify all the finite-dimensional irreducible representations (irreps) of $U_{q_1,q_2}(sl(2/1))$ when both q_1 and q_2 are regarded as indeterminates, and also construct the irreducible cyclic representations when both q_1 and q_2 are roots of unity.

The organization of this paper is as follows. In section 2 we present the two-parameter quantum superalgebra $U_{q_1,q_2}(sl(2/1))$, and study its relationship with the standard quantum supergroup $U_q(sl(2/1))$. In section 3 we classify the finite-dimensional irreps of $U_{q_1,q_2}(sl(2/1))$ when at least one of the deformation parameters is generic, while in section 4 we analyse the properties of $U_{q_1,q_2}(sl(2/1))$ when both q_1 and q_2 are roots of unity, and also construct all its irreducible representations.

2. A two-parameter quantization of $sl(2/1)$

Recall that the Lie superalgebra $sl(2/1)$ is of rank 2. By introducing the three-dimensional vector space with a basis $\{\epsilon_i | i = 1, 2, 3\}$, and defining an inner product $(\,,\,)$ on it such that

$$(\epsilon_i, \epsilon_j) = (-1)^{\delta_{i3}}\delta_{ij}$$

the roots of $sl(2/1)$ can be expressed as

$$\pm(\epsilon_i - \epsilon_j) \qquad i < j.$$

We will denote by H^* the two-dimensional subspace spanned by the roots. There are two Weyl group inequivalent simple root systems. The standard one is $\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}$, while the other has the simple roots

$$\alpha_1 = \epsilon_3 - \epsilon_1 \qquad \alpha_2 = \epsilon_2 - \epsilon_3$$

which obviously satisfy

$$(\alpha_i, \alpha_j) = 1 - \delta_{ij}.$$

With the latter choice for the simple roots, the enveloping algebra $U(sl(2/1))$ of the Lie superalgebra $sl(2/1)$ can be defined as the $\mathbb{Z}_2$-graded associative algebra generated by $\{e_i^{(0)}, f_i^{(0)}, h_i^{(0)} | i = 1, 2\}$, with the relations

$$
\begin{aligned}
&\{e_i^{(0)}, f_j^{(0)}\} = \delta_{ij} h_i^{(0)} \\
&[h_i^{(0)}, e_j^{(0)}] = (\alpha_i, \alpha_j) e_j^{(0)} \\
&[h_i^{(0)}, f_j^{(0)}] = -(\alpha_i, \alpha_j) f_j^{(0)} \\
&[h_i^{(0)}, h_j^{(0)}] = 0 \\
&(e_i^{(0)})^2 = (f_i^{(0)})^2 = 0 \qquad \forall i, j
\end{aligned}
\tag{1}
$$

where

$$\{x, y\} = xy + yx \qquad [x, y] = xy - yx$$

and the gradation is defined by

$$\deg(e_i^{(0)}) = \deg(f_i^{(0)}) = 1 \qquad \deg(h_i^{(0)}) = 0 \qquad \forall i.$$

An important feature of the above definition is that no Serre relations appear explicitly. The nilpotency of $e_i^{(0)}$ and $f_i^{(0)}$, for $i = 1, 2$, guarantees the finite-dimensionality of $sl(2/1)$. Our purpose in this section is to quantize the enveloping algebra $U(sl(2/1))$ in the most general way, requiring that the resultant quantum superalgebra has non-trivial finite-dimensional irreducible representations.

It is straightforward to deform $U(sl(2/1))$ with one parameter. However, a two-parameter quantization can also be obtained, which we denote by $U_{q_1,q_2}(sl(2/1))$. It is a $\mathbb{Z}_2$-graded associative algebra generated by $\{e_i, f_i, h_i | i = 1, 2\}$, with the relations

$$
\begin{aligned}
&\{e_i, f_j\} = \delta_{ij} \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}} \\
&q_i^{\pm h_i} e_j q_i^{\mp h_i} = q_i^{\pm(\alpha_i, \alpha_j)} e_j \\
&q_i^{\pm h_i} f_j q_i^{\mp h_i} = q_i^{\mp(\alpha_i, \alpha_j)} f_j \\
&q_i^{\pm h_i} q_j^{\pm h_j} = q_j^{\pm h_j} q_i^{\pm h_i} \\
&q_i^{h_i} q_i^{-h_i} = 1 \\
&(e_i)^2 = (f_i)^2 = 0 \qquad \forall i, j
\end{aligned}
\tag{2}
$$

and the grading defined by

$$\deg(e_i) = \deg(f_i) = 1 \qquad \deg(q_i^{\pm h_i}) = 0 \qquad \forall i.$$

We wish to emphasize that it is not possible to eliminate any one of the two parameters by, for example, redefinition of the generators etc. Therefore this quantum superalgebra indeed depends on q_1 and q_2 intrinsically. It also possesses non-trivial finite-dimensional irreducible representations when both of the deformation parameters are regarded as indeterminates, as we will show in the next section. Recall that in the standard choice of simple roots, no such multiparameter quantization is allowed, as it would in general spoil the Serre relations, and this in turn rules out the possibility of having any non-trivial finite-dimensional representations.

When $q_1 = q_2 = q$, the above superalgebra reduces to $U_{q,q}(sl(2/1))$, which admits the following comultiplication

$$\Delta(e_i) = e_i \otimes q^{h_i} + 1 \otimes e_i$$

$$\Delta(f_i) = f_i \otimes 1 + q^{-h_i} \otimes f_i$$

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i \qquad \forall i.$$

Counit and antipode can also be introduced in this special case, hence, $U_{q,q}(sl(2/1))$ has the structure of a $\mathbb{Z}_2$-graded Hopf algebra. It should be pointed out that the defining relations for the Lie superalgebra $sl(2/1)$ and its one-parameter deformation $U_{q,q}(sl(2/1))$ have also been obtained in [14].

An important question is how $U_{q,q}(sl(2/1))$ relates to the standard quantum supergroup $U_q(sl(2/1))$. For the sake of concreteness, we copy the definition of the latter below. It is a $\mathbb{Z}_2$-graded Hopf algebra generated by $\{E_i, F_i, H_i | i = 1, 2\}$ with the relations

$$[E_i, F_j\} = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}} \qquad [H_i, H_j] = 0$$

$$[H_i, E_j] = a_{ij} E_j \qquad [H_i, F_j] = -a_{ij} F_j$$

$$(E_2)^2 = (F_2)^2 = 0$$

$$(E_1)^2 E_2 - (q + q^{-1}) E_1 E_2 E_1 + E_2 (E_1)^2 = 0$$

$$(F_1)^2 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_2 (F_1)^2 = 0$$

where

$$(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

and the grading is defined by

$$\deg(E_1) = \deg(F_1) = \deg(H_i) = 0$$

$$\deg(E_2) = \deg(F_2) = 1.$$

A comultiplication is given by

$$\Delta(E_i) = E_i \otimes q^{H_i} + 1 \otimes E_i$$

$$\Delta(F_i) = F_i \otimes 1 + q^{-H_i} \otimes F_i$$

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \qquad \forall i.$$

Let

$$E_0 = q^{H_1+3H_2}(F_2F_1 - qF_1F_2)$$
$$F_0 = (E_1E_2 - q^{-1}E_2E_1)q^{-(H_1+3H_2)}$$

and define a map $U_q(sl(2/1)) \to U_{q,q}(sl(2/1))$ by

$$E_0 \mapsto e_1 \qquad F_0 \mapsto f_1$$
$$E_2 \mapsto e_2 \qquad F_2 \mapsto f_2$$
$$H_2 \mapsto h_2 \qquad H_1 \mapsto -h_1 - h_2.$$

Using lemma 4 of [15] we can easily show that this map gives rise to an algebra isomorphism, which, however, does not preserve the coalgebraic structures. Other algebra isomorphisms between $U_q(sl(2/1))$ and $U_{q,q}(sl(2/1))$ can also be constructed, but again none of them qualifies as a Hopf algebra map. Therefore we conclude that $U_q(sl(2/1))$ and $U_{q,q}(sl(2/1))$ are inequivalent $\mathbb{Z}_2$-graded Hopf algebras. This fact is interesting, as one may construct inequivalent solutions of the Yang–Baxter equation by using the same representation, but different coalgebraic structures.

Below we give the universal R-matrix for $U_{q,q}(sl(2/1))$. Define

$$e = e_1e_2 + q^{-1}e_2e_1 \tag{3}$$
$$f = -(f_2f_1 + qf_1f_2) \tag{4}$$

we have

$$R = q^{h_1\otimes h_2 + h_2\otimes h_1}\tilde{R}$$
$$\tilde{R} = (1\otimes 1 - e_1\otimes f_1)\sum_{n=0}^{\infty}\frac{(q-q^{-1})^n}{(n)_q!}(e\otimes f)^n(1\otimes 1 - e_2\otimes f_2) \tag{5}$$

where

$$(n)_q! = \begin{cases} \prod_{i=1}^{n} q^{1-i}[i]_q & n>0 \\ 1 & n=0 \end{cases}$$

with

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

It is tedious but not very difficult to prove by direct computation that the R defined above indeed satisfies all the defining relations of a universal R-matrix.

It is worthwhile finding out whether $U_{q_1,q_2}(sl(2/1))$ is a Hopf superalgebra when $q_1 \neq q_2$, and if so, whether it is quasitriangular, i.e. admits a universal R-matrix. However, even if the answers to these questions are negative, $U_{q_1,q_2}(sl(2/1))$ at $q_1 \neq q_2$ is still interesting, and well merits a thorough study. This quantum superalgebra has the important properties that the BPW theorem for $U(sl(2/1))$ is preserved (see section 3), and all but one family of the finite-dimensional irreps of $U(sl(2/1))$ are deformed to finite-dimensional irreps of $U_{q_1,q_2}(sl(2/1))$. No such multiparameter quantization appears to exist for ordinary Lie algebras, e.g. $sl(3)$, or even the Lie superalgebra $sl(2/1)$ itself with the standard simple root system. Also, from the point of view of representation theory, this quantum superalgebra is particularly interesting. As will be discussed in detail in the next section, $U_{q_1,q_2}(sl(2/1))$ does not contain any non-commutative even quantum subgroup when $q_1 \neq q_2$, thus the well developed induced module construction for quantum supergroup representations cannot be applied to this quantum superalgebra, and a new method has to be devised in order to construct its representations systematically.

3. Highest-weight representations

In this section we classify the finite-dimensional irreducible representations of $U_{q_1,q_2}(sl(2/1))$ when both q_1 and q_2 are regarded as indeterminates, or one of them is a root of unity. The remaining case will be treated in the next section. Note that when both q_1 and q_2 are indeterminates, $U_{q_1,q_2}(sl(2/1))$ does not have a quantum $sl(2)$ subalgebra, thus the method developed in [16] for constructing highest-weight irreps for type I quantum supergroups does not apply here, and new techniques are required.

Define

$$N_+ = \{(e_1e_2)^k,\ e_2(e_1e_2)^k,\ (e_2e_1)^k,\ e_1(e_2e_1)^k | k = 0, 1, \ldots\}$$

$$N_0 = \{(h_1)^k(h_2)^l | k, l = 0, 1, \ldots\}$$

$$N_- = \{(f_1f_2)^k,\ f_2(f_1f_2)^k,\ (f_2f_1)^k,\ f_1(f_2f_1)^k | k = 0, 1, \ldots\}.$$

Then

$$U_{q_1,q_2}(sl(2/1)) = N_-N_0N_+.$$

To construct representations for this algebra, we consider a one-dimensional N_0N_+-module $\{v^\Lambda\}$ such that

$$e_iv^\Lambda = 0 \qquad h_iv^\Lambda = (\Lambda, \alpha_i)v^\Lambda \qquad \Lambda \in H^*$$

where H^* and the bilinear form (,) on it are as defined at the beginning of the last section. Now we construct the $U_{q_1,q_2}(sl(2/1))$-module

$$\bar{V}(\Lambda) = N_-\{v^\Lambda\}$$

which, however, may not be irreducible in general, and in that case, contains a unique maximal proper submodule $M(\Lambda)$. It is obvious that the quotient module

$$V(\Lambda) = \bar{V}(\Lambda)/M(\Lambda)$$

is irreducible and, as will become clear later, uniquely characterized by Λ. Our purpose is to determine the necessary and sufficient conditions on Λ in order for the $U_{q_1,q_2}(sl(2/1))$-module $V(\Lambda)$ to be finite dimensional, and also construct a basis for this module.

Now we assume that *both* q_1 *and* q_2 *are indeterminates*. Let

$$\lambda_i = (\Lambda, \alpha_i) \qquad i = 1, 2$$

then any given Λ must belong to one of the following three cases:

(i) $\lambda_1 = \lambda_2 = 0$

(ii) $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$

(iii) $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

The first case yields the trivial module. In the second case, we first look at the situation $\lambda_1 = 0, \lambda_2 \neq 0$. If $V(\Lambda)$ is finite dimensional, there must exist such a k_Λ that is the smallest positive integer rendering $f_2(f_1 f_2)^{k_\Lambda} v^\Lambda = 0$. Therefore

$$e_1 e_2 f_2(f_1 f_2)^{k_\Lambda} v^\Lambda = [\lambda_2 - k_\Lambda]_{q_2}[-k_\Lambda]_{q_1} f_2(f_1 f_2)^{k_\Lambda - 1} v^\Lambda = 0$$

which requires $k_\Lambda = \lambda_2 \in \mathbb{Z}_+$. The situation with $\lambda_1 \neq 0$, $\lambda_2 = 0$, can be studied similarly, and the dimensions and bases for the irreducible modules can also be easily obtained. We have the following results:

$$\begin{aligned} &\lambda_1 = 0 \qquad \lambda_2 \in \mathbb{Z}_+ \\ &V(\Lambda) = \{v^\Lambda,\ (f_1 f_2)^{k+1} v^\Lambda,\ f_2(f_1 f_2)^k v^\Lambda | k = 0, 1, \ldots, \lambda_2 - 1\} \\ &\dim V(\Lambda) = 2\lambda_2 + 1 \end{aligned} \tag{6}$$

$$\begin{aligned} &\lambda_1 \in \mathbb{Z}_+ \qquad \lambda_2 = 0 \\ &V(\Lambda) = \{v^\Lambda,\ (f_2 f_1)^{k+1} v^\Lambda,\ f_1(f_2 f_1)^k v^\Lambda | k = 0, 1, \ldots, \lambda_1 - 1\} \\ &\dim V(\Lambda) = 2\lambda_1 + 1. \end{aligned} \tag{7}$$

Now we study the case with $\lambda_1 \neq 0$, $\lambda_2 \neq 0$. We again denote by $V(\Lambda)$ the irreducible $U_{q_1,q_2}(sl(2/1))$-module with highest-weight Λ, and the highest-weight vector v^Λ. In order for $V(\Lambda)$ to be finite dimensional, there must exist positive integers k_Λ and l_Λ satisfying

$$f_2(f_1 f_2)^{k_\Lambda} v^\Lambda = f_1(f_2 f_1)^{l_\Lambda} v^\Lambda = 0.$$

We assume that k_Λ and l_Λ are the smallest positive integers having this property, then the equations

$$e_1 e_2 f_2(f_1 f_2)^{k_\Lambda} v^\Lambda = ([k_\Lambda - \lambda_1]_{q_1}[k_\Lambda - \lambda_2]_{q_2} - [\lambda_1]_{q_1}[\lambda_2]_{q_2}) f_2(f_1 f_2)^{k_\Lambda - 1} v^\Lambda = 0$$

$$e_2 e_1 f_1(f_2 f_1)^{l_\Lambda} v^\Lambda = ([l_\Lambda - \lambda_1]_{q_1}[l_\Lambda - \lambda_2]_{q_2} - [\lambda_1]_{q_1}[\lambda_2]_{q_2}) f_1(f_2 f_1)^{l_\Lambda - 1} v^\Lambda = 0$$

require that

$$\begin{aligned} &[k_\Lambda - \lambda_1]_{q_1}[k_\Lambda - \lambda_2]_{q_2} = [\lambda_1]_{q_1}[\lambda_2]_{q_2} \\ &[l_\Lambda - \lambda_1]_{q_1}[l_\Lambda - \lambda_2]_{q_2} = [\lambda_1]_{q_1}[\lambda_2]_{q_2}. \end{aligned} \tag{8}$$

When both q_1 and q_2 are regarded as indeterminates, this set of simultaneous equations admits positive integer solutions for k_Λ and l_Λ if and only if

$$\lambda_1 = \lambda_2 = \lambda \neq 0 \qquad 2\lambda \in \mathbb{Z}_+ \tag{9}$$

and the corresponding solutions read

$$k_\Lambda = l_\Lambda = 2\lambda.$$

To see how (9) arises, consider, say, the first equation of (8). Take the limit $q_1, q_2 \to 1$; we obtain

$$k_\Lambda = \lambda_1 + \lambda_2.$$

Putting this back into the original equation, we immediately see that $\lambda_1 = \lambda_2$ is also required. Needless to say, one naturally expects constraints like (9) on the highest weight, just as in the case of classical $su(2)$, only integer and half-integer spin representations are of finite dimension.

To construct a basis for the irreducible $U_{q_1,q_2}(sl(2/1))$-module $V(\Lambda)$ with Λ satisfying (9), we note that

$$(f_1 f_2)^{2\lambda} v^\Lambda = (f_2 f_1)^{2\lambda} v^\Lambda$$

hence we have

$$\begin{aligned} V(\Lambda) &= \{(f_1 f_2)^k v^\Lambda,\ f_2(f_1 f_2)^k v^\Lambda,\ (f_2 f_1)^{k+1} v^\Lambda,\ f_1(f_2 f_1)^k v^\Lambda | k = 0, 1, \ldots, 2\lambda - 1\} \\ \dim V(\Lambda) &= 8\lambda. \end{aligned} \tag{10}$$

However, when $q_1 = q_2$, the more general condition

$$\lambda_1 + \lambda_2 \in \mathbb{Z}_+ \qquad \lambda_1 \neq 0,\ \lambda_2 \neq 0 \tag{11}$$

is necessary and sufficient to insure the existence of positive integer solutions of (8)

$$k_\Lambda = l_\Lambda = \lambda_1 + \lambda_2.$$

In this case, a basis for $V(\Lambda)$ reads

$$\begin{aligned} V(\Lambda) &= \{(f_1 f_2)^k v^\Lambda,\ f_2(f_1 f_2)^k v^\Lambda,\ (f_2 f_1)^{k+1} v^\Lambda,\ f_1(f_2 f_1)^k v^\Lambda | k = 0, 1, \ldots, \lambda_1 + \lambda_2 - 1\} \\ \dim V(\Lambda) &= 4(\lambda_1 + \lambda_2). \end{aligned} \tag{12}$$

It is clear that the irreps constructed above constitute all the finite-dimensional irreps of the quantum superalgebra $U_{q_1,q_2}(sl(2/1))$ when both q_1 and q_2 are regarded as indeterminates. As $U_{q,q}(sl(2/1))$ is algebraically isomorphic to the standard quantum supergroup $U_q(sl(2/1))$, there should exist a one-to-one correspondence between the irreps constructed here when $q_1 = q_2$ and those of [16]. This is indeed the case: the irreps with one of the λ's vanishing are equivalent to the atypicals, and those with both λ's non-vanishing to the typicals [16]. Note that in the latter case, both λ_1 and λ_2 can be complex numbers, so long as their sum is a positive integer. This accounts for the one complex parameter family of finite-dimensional typical irreps obtained in [16].

Also observe that the finite-dimensionality condition (9) is so stringent that it excludes the highest weights where both λ_1 and λ_2 are arbitrary positive integers. This might be an indication that, when $q_1 \neq q_2$, $U_{q_1,q_2}(sl(2/1))$ does not admit a co-multiplication, as otherwise it would be conceivable that repeated tensor products of the irreps given in (6) and (7) could yield finite-dimensional irreps with such highest weights. (Of course we should be aware that finite-dimensional representations of $U_{q_1,q_2}(sl(2/1))$ are not fully reducible, thus the reliability of the above argument needs investigation.)

We now turn to the construction of highest-weight irreps of $U_{q_1,q_2}(sl(2/1))$ when one of the deformation parameters is a root of unity, while the other is regarded as an indeterminate. We should point out that in the present case, if one disregards the h_i's in $U_{q_1,q_2}(sl(2/1))$ and considers the $q_i^{-h_i}$'s only, then several highest weights may lead to the same irrep. A detailed discussion of this problem will be given in the next section, here we merely construct the irreps.

We assume that q_1 is generic, but q_2 is a root of unity. Let N' be the smallest positive integer such that $q_2^{N'} = 1$. Define

$$N = \begin{cases} N'/2 & \text{if } N' \text{ is even} \\ N' & \text{if } N' \text{ is odd.} \end{cases}$$

There exists three classes of finite-dimensional irreps of $U_{q_1,q_2}(sl(2/1))$ with the highest weights, respectively, satisfying the following conditions:

(i) $\lambda_1 \in \mathbb{Z}_+$, $\lambda_2 \equiv 0 \pmod N$.
A basis and the dimension for the finite-dimensional irreducible $U_{q_1,q_2}(sl(2/1))$-module $V(\Lambda)$ are given by

$$\begin{aligned} &V(\Lambda) = \{v^\Lambda,\ (f_2 f_1)^{k+1} v^\Lambda,\ f_1 (f_2 f_1)^k v^\Lambda \mid k = 0, 1, \ldots, \lambda_1 - 1\} \\ &\dim V(\Lambda) = 2\lambda_1 + 1. \end{aligned} \tag{13}$$

(ii) $\lambda_1 = 0$, $\lambda_2 \in \mathbb{Z}_+$.
In this case we define $\tilde{\lambda}_2 \in \{0, 1, \ldots, N-1\}$ by $\lambda_2 \equiv \tilde{\lambda}_2 \pmod N$, then

$$\begin{aligned} &V(\Lambda) = \{v^\Lambda,\ (f_1 f_2)^{k+1} v^\Lambda,\ f_2 (f_1 f_2)^k v^\Lambda \mid k = 0, 1, \ldots, \tilde{\lambda}_2 - 1\} \\ &\dim V(\Lambda) = 2\tilde{\lambda}_2 + 1. \end{aligned} \tag{14}$$

(iii) $\lambda_1 \neq 0$, $\lambda_2 \not\equiv 0 \pmod N$.
In this case there are two possibilities for finite-dimensional irreps to exist. One possibility is

$$2\lambda_1 \in \mathbb{Z}_+ \qquad 2\lambda_2 \equiv 2\lambda_1 \pmod{N'}$$

and the other is

$$2\lambda_1 \in \mathbb{Z}_+ \qquad 2\lambda_1 \equiv N \pmod{N'} \qquad N' \text{ even} \qquad \lambda_2 \in \mathbb{C}.$$

In both cases, we have

$$\begin{aligned} &V(\Lambda) = \{(f_1 f_2)^k v^\Lambda,\ f_2 (f_1 f_2)^k v^\Lambda,\ (f_2 f_1)^{k+1} v^\Lambda,\ f_1 (f_2 f_1)^k v^\Lambda \mid k = 0, 1, \ldots, 2\lambda_1 - 1\} \\ &\dim V(\Lambda) = 8\lambda_1. \end{aligned} \tag{15}$$

As the roles of q_1 and q_2 can be interchanged by exchanging the indices 1 and 2 for the generators, the classification of irreps in the case when q_2 is generic and q_1 is a root of unity can be easily derived from the results obtained above.

4. Cyclic representations

In order to develop the representation theory for $U_{q_1,q_2}(sl(2/1))$ when both of the deformation parameters are roots of unity, we alter the definition of the quantum superalgebra slightly by saying that it is generated by $\{e_i,\ f_i,\ q_i^{\pm h_i}|i = 1,2\}$ subject to the relations (2). Properties of $U_{q_1,q_2}(sl(2/1))$ in the present case differ drastically from those where at least one of the deformation parameters is generic. Let N_1' and N_2' be the smallest positive integers such that $q_i^{N_i'} = 1,\ i = 1,2$. Define

$$N_i = \begin{cases} N_i'/2 & \text{if } N_i' \text{ is even} \\ N_i' & \text{if } N_i' \text{ is odd} \end{cases}$$

and let M be the smallest positive integer divisible by both N_1' and N_2' and

$$M' = \begin{cases} M/2 & N_i' \text{ even, } M/2 \text{ odd} \\ M & \text{otherwise.} \end{cases}$$

We have the following useful result: the elements

$$\begin{aligned} \Gamma_+ &= (e_1e_2)^M + (e_2e_1)^M \\ \Gamma_- &= (f_1f_2)^M + (f_2f_1)^M \\ z_i^{\pm 1} &= (q_i^{\pm h_i})^M \qquad i = 1,2 \end{aligned} \tag{16}$$

all commute with $U_{q_1,q_2}(sl(2/1))$. We denote by Z_0 the algebra generated by $\Gamma_\pm$ and $z_i^{\pm 1}$. Then it is obvious that Z_0 does not have zero divisors, i.e. for any $x, y \in Z_0$, if $x \neq 0,\ y \neq 0$, then $xy \neq 0$. Therefore it makes sense to talk about the quotient field of Z_0, which we denote by $Q(Z_0)$. Over this field, $U_{q_1,q_2}(sl(2/1))$ is finite dimensional. More explicitly, letting

$$Q(U_{q_1,q_2}(sl(2/1))) = Q(Z_0) \otimes_{Z_0} U_{q_1,q_2}(sl(2/1))$$

we have

$$Q(U_{q_1,q_2}(sl(2/1))) = Q(N_+)Q(N_0)Q(N_-)$$

with

$$\begin{aligned} Q(N_+) &= \{(e_1e_2)^k,\ e_2(e_1e_2)^k,\ (e_2e_1)^k,\ e_1(e_2e_1)^k | k = 0,1,\ldots,M-1\} \\ Q(N_0) &= \{(q_1^{h_1})^k (q_2^{h_2})^l | k,l = 0,1,\ldots,M-1\} \\ Q(N_-) &= \{(f_1f_2)^k,\ f_2(f_1f_2)^k,\ (f_2f_1)^k,\ f_1(f_2f_1)^k | k = 0,1,\ldots,M-1\}. \end{aligned}$$

In plain terms, the quantum superalgebra $U_{q_1,q_2}(sl(2/1))$ is finite dimensional, up to the elements of Z_0. Therefore, all its irreps must also be finite dimensional.

Let us first consider the highest-weight irreps. The highest-weight vector v^Λ of the irreducible $U_{q_1,q_2}(sl(2/1))$-module $V(\Lambda)$ satisfies, by definition,

$$e_i v^\Lambda = 0 \qquad q_i^{h_i} v^\Lambda = q_i^{\lambda_i} v^\Lambda \qquad \forall i.$$

Note that the λ_i's are not uniquely defined, as $q_i^{\lambda_i+kN_i'}=q_i^{\lambda_i}$. To eliminate this arbitrariness, we require that

$$0\leqslant \operatorname{Re}\lambda_i<N_i \qquad i=1,2 \tag{17}$$

where $\operatorname{Re}\lambda_i$ denotes the real part of λ_i. Then each irrep is uniquely characterized by such a $\Lambda\in H^*$. (One may think that (17) is too restrictive to allow for all the irreps, as $q_i^{N_i}=-1$ if N_i' is even. This of course is not true, as the sign of $q_i^{h_i}$ can be altered by the isomorphisms $e_i\mapsto\omega_i e_i,\ f_i\mapsto f_i,\ q_i^{h_i}\mapsto\omega_i q_i^{h_i},\ \omega_i=\pm1,\ i=1,2.$)

The highest weights may be classified into three types; we explicitly construct the bases for the irreducible highest-weight $U_{q_1,q_2}(sl(2/1))$-modules in each type.

(i) $\lambda_1=\lambda_2=0$:
We have the trivial irrep.

(ii) One of λ_1 and λ_2 is non-vanishing:
Define $\mathbb{Z}_K=\{0,1,\ldots,K-1\},\ 0<K\in\mathbb{Z}_+$, then

$$\begin{aligned}&\lambda_1=0\\ &V(\Lambda)=\{v^\Lambda,\ (f_1f_2)^{k+1}v^\Lambda,\ f_2(f_1f_2)^k v^\Lambda|k=0,1,\ldots,(d-3)/2\}\\ &\dim V(\Lambda)=d=\begin{cases}2\lambda_2+1 & \lambda_2\in\mathbb{Z}_{N_2}\\ 2N_2+1 & \lambda_2\notin\mathbb{Z}_{N_2}\end{cases}\end{aligned} \tag{18}$$

$$\begin{aligned}&\lambda_2=0\\ &V(\Lambda)=\{v^\Lambda,\ (f_2f_1)^{k+1}v^\Lambda,\ f_1(f_2f_1)^k v^\Lambda|k=0,1,\ldots,(d-3)/2\}\\ &\dim V(\Lambda)=d=\begin{cases}2\lambda_1+1 & \lambda_1\in\mathbb{Z}_{N_1}\\ 2N_1+1 & \lambda_1\notin\mathbb{Z}_{N_1}.\end{cases}\end{aligned} \tag{19}$$

(iii) Both λ_1 and λ_2 are non-vanishing:

$$\begin{aligned}&V(\Lambda)=\{v^\Lambda,\ (f_2f_1)^{k+1}v^\Lambda,\ f_1(f_2f_1)^k v^\Lambda|k=0,1,\ldots,d/4-1\}\\ &\dim V(\Lambda)=d=\begin{cases}4(\lambda_1+\lambda_2) & N_1\neq N_2,\ \lambda_1=\lambda_2,\ \lambda_1+\lambda_2\in\mathbb{Z}_M\\ 4k_\Lambda & N_1=N_2,\ \lambda_1+\lambda_2\in\mathbb{Z}_+\\ 4M & \text{otherwise}\end{cases}\end{aligned} \tag{20}$$

where k_Λ is defined by $N_1>k_\Lambda\in\mathbb{Z}_+$, and $k_\Lambda\equiv\lambda_1+\lambda_2 \pmod{N_1}$.

A common feature of the irreducible highest-weight $U_{q_1,q_2}(sl(2/1))$-modules constructed above is that the invariants $\Gamma_\pm$ both take zero eigenvalue. This distinguishes them from a class of irreps, called cyclic, where we have non-vanishing Γ_- and/or Γ_+.

To construct the cyclic irreps, we start with a one-dimensional module $\{v_0\}$ over the superalgebra generated by $\{e_1,\ f_2,\ q_i^{\pm h_i}\}$ such that

$$e_1v_0=f_2v_0=0 \qquad q_i^{\pm h_i}v_0=q_i^{\pm\mu_i}v_0 \qquad i=1,2. \tag{21}$$

Define

$$V_0 = \{v_k = (e_1e_2)^k v_0 | k = 0, 1, \ldots, M-1\}$$
$$\bar{V} = V_0 \oplus f_1 V_0 \oplus e_2 V_0 \oplus f_1 e_2 V_0. \tag{22}$$

We specify the action of $f_2 f_1$ and the eigenvalue of Γ_+ in $\bar{V}$ by

$$\Gamma_+ v_0 = \gamma_+ v_0 \qquad \gamma_+ \in \mathbb{C}$$
$$f_2 f_1 v_0 = \chi (e_1 e_2)^{M-1} v_0 = \chi v_{M-1}. \tag{23}$$

Then $\bar{V}$ gives rise to a $U_{q_1,q_2}(sl(2/1))$-module, which, although obviously indecomposible, may not be irreducible in general, and in that case we factor out from it its maximal proper submodule I to obtain the irreducible $U_{q_1,q_2}(sl(2/1))$-module

$$V = \bar{V}/I.$$

Since Γ_- commutes with all the elements of $U_{q_1,q_2}(sl(2/1))$, it takes the constant eigenvalue γ_- on $\bar{V}$, which can be easily worked out to be

$$\gamma_- = \chi \prod_{k=1}^{M-1} ([\mu_1 + k]_{q_1}[\mu_2 + k - 1]_{q_2} - [\mu_1]_{q_1}[\mu_2 - 1]_{q_2} + \chi\gamma_+).$$

Now several remarks are in order: note that the requirement $e_1 v_0 = f_2 v_0 = 0$ does not impose any constraint on the irreducible module V. As both e_1 and f_2 are nilpotent and anticommute with each other, there always exist vectors satisfying the condition in any irreducible module. Also, the first equation of (23) holds in any irrep with γ_+ a complex parameter determined by the irrep itself. However, the second equation of (23) is necessary in order to turn $\bar{V}$ into a $U_{q_1,q_2}(sl(2/1))$-module. Also, for $v_k \in V_0$

$$e_1 v_k = f_2 v_k = 0$$
$$f_2 f_1 v_k = ([\mu_1 + k]_{q_1}[\mu_2 + k - 1]_{q_2} - [\mu_1]_{q_1}[\mu_2 - 1]_{q_2} + \chi\gamma_+) v_{k-1}$$

therefore, every v_k is on the same footing as v_0. We require that the $\gamma_\pm$ do not vanish simultaneously, as otherwise V would be isomorphic to one of the highest-weight modules discussed earlier. From now on we assume that this condition is fulfilled. Observe that

$$f_2 e_1 f_1 e_2 v_k = \delta v_k \qquad \delta \doteq [\mu_1]_{q_1}[\mu_2 - 1]_{q_2} - \chi\gamma_+.$$

If $\delta \neq 0$, then $\bar{V}$ itself is irreducible, i.e. $V = \bar{V}$. In fact, in the special case with $N_1' = N_2'$, V coincides with the typical cyclic irreducible $U_q(sl(2/1))$-module constructed in [16]. For this reason, we call V typical.

When $\delta = 0$ we call V atypical. In this case

$$I = \{f_1 e_2 v_k,\ w_k = [\mu_1 + k]_{q_1} e_2 v_k - f_1 v_{k+1} | k = 0, 1, \ldots, M-1\}.$$

Hence

$$V = V_0 \oplus V_1 \qquad V_1 = (f_1 V_0 \oplus e_2 V_0)/\{w_k | k = 0, 1, \ldots, M-1\}.$$

The matrix elements of the generators of $U_{q_1,q_2}(sl(2/1))$ in the cyclic irreps can also be easily worked out, but we will not do it here.

5. Discussion

We have shown that the one-parameter quantum supergroup $U_{q,q}(sl(2/1))$ associated with $sl(2/1)$ in its non-standard simple root system is algebraically equivalent to the standard quantum supergroup $U_q(sl(2/1))$. However, it remains to be seen whether this is also true for the other superalgebras, and it might turn out that the answer to this question is negative in general.

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